

## The Contest As a System of Balanced Forces

The form of the contest equations can be explained by means of a system in which driving forces are opposed by frictional forces. Let us start by picturing a two-person contest as a situation in which each person has a body which he is trying to move with his full strength. But each body has a brake pad beside it, with the brake on body 1 activated by person 2 and vice versa. Think of the person trying to move the body as the *attacker* and the person using his full strength in trying to keep the body from moving as the *defender* in the contest. The strengths of the contestants (designated  $X_1$  and  $X_2$ ) are the maximum forces they can apply to their attack and defense roles.

The next bit of conceptual apparatus needed for this derivation is a lever arm of length  $L$  for each participant, with each arm attached to a rigid structure at one end, but free to pivot (like a pry bar). The contest also needs a reference force,  $F$ , whose purpose will become clear later and whose strength need not be specified (except to say that it is the same for both contestants). Now, contestant 1 determines that he can use his strength at a given point on his lever, at a distance  $L$  from the pivot end, to produce a force  $F$  at a distance  $a_1$  from the pivot end (as in lifting a rock). This means that  $X_1 = F (a_1/L)$  and contestant 2 can do the same thing, using the same distance  $L$ , to get a similar relationship between his strength and a distance  $a_2$  on his lever. The distances  $L$ ,  $a_1$ , and  $a_2$  can be marked on the respective lever arms. What we have done is give each contestant a calibrated lever.

Next, assume that the two brakes have friction material with the same coefficient of friction; call it  $\mu$ . (We will not distinguish between the static and dynamic coefficient of friction.) The drag force on body 1 due to the brake force applied by contestant 2 is  $\mu X_2$ , and that drag force is just the amount required to stop the motion of the body when it is pushed by contestant 1. But there must be some method of adjustment to achieve the exact stopping condition, and the lever of contestant 1 gives the means for adjusting. Suppose contestant 1 uses his strength at some adjustable point on the lever and the force at point  $L$  is used to move body 1. After sliding the point of contact along the lever, it is found that the body stops when that point of contact is at a distance  $b_{12}$  from the pivot. The driving force on the body,  $X_1 (b_{12}/L)$ , will be equal to the drag force produced by the brake so

$$\mu X_2 = X_1 (b_{12}/L).$$

But we found earlier that  $X_1 = F (a_1/L)$  so we can eliminate  $L$  and obtain

$$\mu X_2 = (X_1)^2 (b_{12}/a_1)/F$$

or

$$(X_1)^2 = [\mu (a_1/b_{12}) F] X_2 .$$

The expression in square brackets is a “given” in the problem--a scoreboard entry for attacker 1 against defender 2; we need have no concern about the items inside the brackets. Of course, a similar equation will apply for attacker 2 against defender 1. We can now write the two equations for this contest as

$$(X_1)^2 = A_{12} X_2$$

$$(X_2)^2 = A_{21} X_1$$

where  $A_{12}$  and  $A_{21}$  are scoreboard entries. Because this mechanical analog was set up as a force-balance problem, the two constants have the dimension of a force. But a contest can have scoreboard entries with any sort of dimension--dollars or hours, for example--or no dimension at all. This means that the dimension of the strengths,  $X_i$ , must be the same as whatever dimension the scoreboard entries have. It is useful to think of the strength as a force, but it is by no means necessary. Sometimes it is helpful to consider the strength as an *importance* with no particular dimension.

A feature of this model is the possibility for self adjustment or “automatic tuning”. By connecting a cable via pulleys from the movable body to the sliding point on the lever where the attacking force is applied, the motion of the body can be used to slide the point closer to the pivot of the lever arm, reducing the driving force on the body. When the driving force is weak enough, the brake will stop the motion at the force-balance point..

An interesting way of looking at the problem is to rewrite the basic equation, dividing through by  $X_1$ , giving  $X_1 = [\mu (a_1/b_{12}) F/X_1] X_2 = [\mu (L/b_{12})]X_2$ . This returns the quantity in square brackets to non-dimensionality and makes it look like an adjusted coefficient of friction, in which the adjustment depends upon the amount by which the lever changes the driving force. In other words, the attacker's force adjustment, to achieve balance, can be construed as an adjustment of the coefficient of friction on the brake. Since the attacker does the adjusting, it is necessary to do the force-balance procedure only once; the defense force applied to the brake does not change. No cycling back and forth is needed to arrive at a final adjustment. The working of the contest algorithm is to take already-adjusted coefficients of friction (as given by the contest scoreboard) and then work backwards to find the strengths of the contestants.

Solving the equations gives

$$X_1 = (A_{12}^2 A_{21})^{1/3} \quad \text{and} \quad X_2 = (A_{21}^2 A_{12})^{1/3}$$

showing that, while most of the exponential emphasis in the strength of a contestant comes from his attack, a significant part of it comes from his ability to defend against the attack of his opponent, and that part depends upon the opponent's strength.

To go from the simple duel to a contest involving more than two participants, I will first give the reasoning which led to the discovery of a form of the contest algorithm which was used by the military for many years. In the analysis of military combat, the results of engagements involving many types of weapons are put into tables of fractional losses of weapon types produced by opposing weapon types--loss-by-cause tables or scoreboards. (*Fractional* losses are used so that all entries have a common dimensionality.) It seemed reasonable to expect that the scoreboard data could be processed in some way to produce numbers showing how the weapon systems in an engagement compare with each other. A system that kills a lot of important opposing systems should be important itself and should give a boost of importance to an opposing system that can kill it. Any process for comparing weapon systems must satisfy the requirement for *groupability*: the idea that splitting or combining groups of identical weapons (making more or fewer groups out of them) must make no difference in the importance that is computed for an individual weapon in any of the groups.

The most obvious approach to finding weapon-comparison numbers was to use a system of linear equations, the classical *Eigenvalue* method. I began trying this around 1968 and thought about it occasionally in connection with work I was doing, but I could not make it work. Others independently thought of the method and had the same lack of success. (It had to do with unreasonableness of

solutions in response to smooth changes in inputs.) Then in the mid-80s I got into a problem area (combat modeling) that really needed weapon comparisons and I tried to find a system of *nonlinear* equations that satisfied the groupability criterion (as the linear system did). To my surprise, I found (by trial and error, not derivation) a family of nonlinear equations that did this, and I accepted the member of the family that agreed with the linear method (described in the next paragraph) for the case of two participants.

The linear method for two participants has  $\lambda X_1 = A_{12} X_2$  and  $\lambda X_2 = A_{21} X_1$ , with  $\lambda$  as the *Eigenvalue*. Only relative X values are obtainable, and their ratio is  $(A_{12}/A_{21})^{1/2}$ . The square of this ratio gives the relative contributions of the two orthogonal X vectors to their vector sum or  $(X_1/X_2)^2 = (A_{12}/A_{21})$ . This agreement between the relative contributions and the ratio of the points in a scoreboard was deemed sufficient reason to say that the linear approach was correct for the duel, even if it could not be extended to more participants. (We are used to thinking of the comparison between players in terms of their scoring against each other.)

Now, the family of nonlinear equations satisfying groupability is as follows:

$$(X_i)^n = \sum_j (A_{ij})^{n-1} f_{ij} X_j$$

where  $f_{ij} = A_{ij} / \sum_i A_{ij}$  and where n is an integer greater than 1. The subscript on the summation symbol shows which index is to be summed, i or j.

Agreement between the linear and nonlinear approaches for the relative-contribution ratio in the duel occurs for the n = 3 member of the nonlinear family, and that agreement has until now dictated the use of the n = 3 member in the equations of the contest algorithm. It was always necessary to assume the presence of some form of force adjustment with gearing or hydraulics in a mechanical-analog model of a contest, but no simple adjustment mechanism presented itself. This led me to reconsider the original choice of n = 3 for the member of the family and the need for agreement with the linear equations for the duel. The way a simple lever can be used for adjustment, as described above, leads me now to conclude that the n = 2 member of the nonlinear family is the correct one.

So, where does that leave us with respect to the ratio of relative contributions to the vector-sum magnitude in a duel? That ratio is now  $(X_1/X_2)^2 = (A_{12}/A_{21})^{2/3}$  and it might be of interest to ponder why it is less extreme than the pure ratio of scores. The answer seems to be that attack alone does not determine how the contestants compare. Bringing in the defense capabilities works against the attack capabilities. Putting in some numbers might be a help here. Suppose the ratio of scores were 8:1. This makes the ratio of contributions 4:1, which might make a better comparison, since the weaker side deserves some credit for its defense against a strong opponent. At any rate, there is no *a priori* reason to expect the ratio of strengths to be equal to the ratio of scores.

Now, having settled upon the correct form of the equations for determining strengths of contest participants, let us derive that set for contests larger than the duel. The general scheme is that of a set of force vectors working in a vector space of n dimensions, where n is the number of participants in the contest. Every participant has two vectors, one for attack and the other for defense, both with the same magnitude but generally different in direction. Attack against another participant means having a component of the attack vector pointing in the his direction; defense against that other participant means having a component of the defense vector pointing in his direction. (When contestant i is attacking contestant j in direction j, contestant j is using brake force perpendicular to direction j, or in direction i. Thus the two engaged contestants point vector components at each other, whether attacking

or defending.) This whole n-dimensional picture is purely a mathematical construct; a machine of this sort is imaginary, of course, but its parts can be oriented in any way in real space. Observe now that the fractions  $f_{ij}$  show the division of the defender's force into the directions of his attackers. Each attacker has a movable body with an accompanying brake, and the defender applies force to each brake in proportion to the attack force on each body. Thus attacker  $i$  gets a defending force from each defender equal to  $f_{ij} X_j$  or  $(A_{ij} / \sum_i A_{ij}) X_j$ , where  $j$  is the index of the defender. All that is needed to go from the duel to the general case is to use the fraction  $f_{ij}$  as the cosine of the angle between the defense vector of contestant  $j$  and the vector-space direction of contestant  $i$ . The set of equations making up the contest algorithm is then the same as the  $n = 2$  member of the family that was found by trial and error to satisfy groupability, namely

$$(X_i)^2 = \sum_j (A_{ij}) f_{ij} X_j \quad \text{where} \quad f_{ij} = A_{ij} / \sum_i A_{ij} .$$

These equations are easy to solve by iteration, even for very large problems.

The basic equation gives the square of the magnitude of the strength vector of a contestant, and each term in the sum is the square of a vector component in one of the directions of the vector space. Therefore we can immediately write the vector form of the equation as

$$\underline{X}_i = \sum_j \{ [(A_{ij}) f_{ij} X_j ]^{1/2} \cdot \underline{u}_j \}$$

where the underline designates a vector quantity and  $\underline{u}_j$  is a unit vector in the  $j$  direction. Both  $i$  and  $j$  range from 1 to the number of contestants in the problem. Solving the set of coupled nonlinear equations which make up the algorithm tells us everything there is to know about the set of vectors--magnitudes, attack directions from the vector form just shown, and defense directions from the fractions  $f_{ij}$ .

We can find the angles which the attack vectors make with each other, and this leads into the subject of *enmity* and *amity* in the contest (amity being mutual support). In a duel, it is obvious that enmity must be 100% and amity zero, and it would follow that the right angle between attack vectors in the duel corresponds to full enmity. Full amity could then be nothing other than a zero angle between attack vectors. The cosine of the angle between attack vectors is a logical designator for degree of amity. Only the attack vectors figure in the specification of amity between participants--or in determinations of relative contributions to the contest of any sort--because defense is passive. Defense is not needed by a contestant that is not scored against (attacked), and the direction of the defense vector is indeterminate for that case.

Two contestants which are identical in their behavior with respect to every other contestant must have vectors that are equal in magnitude and parallel for attack and again for defense. Contestants having proportional scores against each of their opponents (*rows* of the data array *proportional* to each other) will have parallel attack vectors. Identical behavior of two contestants in the defense mode means that their scoreboard entries are identical for each of their attackers, causing  $f_{ij}$  to be identical for each  $i$  (i.e., *columns* of the data array *equal* to each other--not proportional). When attack vectors of two contestants are parallel, and defense vectors also, the two contestants are *groupable* and can be replaced with one of combined strength. (Groupability was the reason for the selection of these contest equations for combat analysis in the beginning.) Contestants with parallel vectors in only one of the two modes will not be groupable.

The last paragraph explains why scores in contests must be in the form of *score per defender* whenever

grouping is a consideration. Under grouping, the scoreboard entries for attack (rows) are added together, and one of the (equal) columns for defense is deleted each time a group is formed from two subgroups. This arrangement was automatically satisfied in the original application of this method (combat modeling), because the scores were fractional losses of combatants--and a fractional loss is the same, no matter how many are in the group. It now is clear that all contest situations require the scores to be on a per-defender basis in order that the columns of the score matrix (the scores indexed to different attackers for particular defenders) do not change when defenders are grouped.

At this point, we have a complete imaginary model constructed solely from physical concepts, with strengths of participants and all angular quantities calculable from scoreboard entries alone. Variables describing the operation are lumped together into the scoreboard entries. No theoretical difficulty arises by resorting to an imaginary machine; indeed, the progress of science is studded with *Gedanken* experiments used to illustrate complex points of theory. It should be pointed out that the imaginary nature of this machine has to do with practical considerations alone and not with operation in n dimensions. The directions of potential motion in the machine are given no particular spatial orientation in the real world.

The proof that the system of equations of this paper constitutes a valid description for a contest is in the physical and mathematical description of the machine itself. The only assumption of the method is that contest scores are analogous to friction constants. This assumption is quite reasonable, from a qualitative standpoint, since a large contest score identifies a weak defender who cannot put much force upon his brake and has to rely upon more effective friction material to stop the attacker.

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