

## The Contest As a System of Balanced Forces

The form of the contest equations can be explained by means of a system in which driving forces are opposed by frictional forces. Let us start by picturing a two-person contest as a situation in which each person has a body which he is trying to move with his full strength. But each body has a brake pad beside it, with the brake on body 1 activated by person 2 and vice versa. Think of the person trying to move the body as the *attacker* and the person using his full strength in trying to keep the body from moving as the *defender* in the contest. The strengths of the contestants (designated  $X_1$  and  $X_2$ ) are the forces they apply in their attack and defense roles.

Each brake pad has a friction constant with a critical value--a value which is just great enough to stop the motion and no greater. I will use  $b_{ij}$  for the friction coefficient, where the first index represents the attacker. For the duel, the force-balance equations are

$$X_1 = b_{12} X_2 \quad \text{and} \quad X_2 = b_{21} X_1 .$$

This relationship only tells us that  $b_{12} b_{21} = 1$  and only gives the ratio of the  $X$  values. We need more information, so let us introduce another force ( $F$ ) to act as a "universal defender", with both  $X$  values being proportional to  $F$ . The constants of proportionality ( $c_1$  and  $c_2$ ) can be thought of as friction coefficients for force balance when the defender in each case has strength  $F$ .

$$X_1 = c_1 F \quad \text{and} \quad X_2 = c_2 F .$$

Multiplying the two equations for  $X_1$  together, and similarly for  $X_2$ , we get the following:

$$(X_1)^2 = [(b_{12})c_1 F] X_2 \quad \text{and} \quad (X_2)^2 = [(b_{21})c_2 F] X_1 .$$

The products inside the square brackets are the *given data* for a contest problem--the scoreboard entries--called  $A_{12}$  and  $A_{21}$  in this case. The multiplication gives the square of the "geometric mean" of two computations, showing that the attack strength needs to match the average of the defense combination from two defenders (the other party in the duel and the universal defender).

Doing this another way will illustrate the logic behind the multiplication of two equations: Suppose we say that the strength of the attack vector for a contestant must be equal to the strength of his defense vector, and that the common strength is then equal to the geometric mean of results from two equations--one for the attack vector and the other for the defense vector. For contestant 1 the two equations are

$$X_1 = c_1 F \quad \text{and} \quad X_1 = X_2 / b_{21} \quad \text{for attack and defense respectively.}$$

The geometric mean of the two equations is  $X_1 = [(c_1/b_{21})FX_2]^{1/2}$ , but  $b_{21} = 1/b_{12}$  so, after squaring the geometric mean, we get

$$(X_1)^2 = [(b_{12})c_1 F] X_2 , \quad \text{the same as before.}$$

The equations to be solved for the duel are then

$$(X_1)^2 = A_{12} X_2 \quad \text{and} \quad (X_2)^2 = A_{21} X_1$$

with the result that

$$X_1 = [(A_{12})^2(A_{21})]^{(1/3)} \quad \text{and} \quad X_2 = [(A_{21})^2(A_{12})]^{(1/3)} \quad \text{and} \quad X_1/X_2 = (A_{12}/A_{21})^{1/3} .$$

While most of the exponential emphasis in the strength of a contestant comes from his attack, a significant part of it comes from his ability to defend against the attack of his opponent, and that part depends upon the opponent's strength.

Because this mechanical analog was set up as a force-balance problem, the scoreboard entry  $A_{ij}$  has the dimension of a force. But a contest can have scoreboard entries with any sort of dimension--dollars or hours, for example--or no dimension at all. This means that the dimension of the strengths must be the same as whatever dimension the scoreboard entries have. It is useful to think of the strength as a force, but it is by no means necessary. Sometimes it is helpful to consider the strength as an *importance* with no particular dimension.

The attack vectors are orthogonal for the duel, with vector 1 pointing in direction 2, etc. Comparison of the two contestants comes from their contributions to the *contest vector*, the vector sum of the two attack vectors. The vector sum for the special case of a duel is

$$[(X_1)^2 + (X_2)^2] / [(X_1)^2 + (X_2)^2]^{(1/2)} = [A_{12} X_2 + A_{21} X_1] / [(X_1)^2 + (X_2)^2]^{(1/2)}$$

and we see that participants 1 and 2 contribute to the contest in proportion to their *weighted scores*,  $(A_{ij}X_j)$ . The strength of the defender is the weight,  $X$ . It will be shown later that this weighting scheme holds for contests with any number of participants.

### Deriving the Contest Equation With a Mechanical Model

The simple derivation given above can be explained further with a conceptual model that shows how the adjustment of the coefficients of friction can be done with hardware. The main bit of apparatus needed is a lever arm for each participant, with each arm free to pivot at one end (like a pry bar). The contest also needs a reference force,  $F$ , whose purpose will become clear later and whose strength need not be specified (except to say that it is the same for both contestants). Each contestant can apply his force anywhere on one side of his lever and the force transmitted to the movable body comes off the other side of the lever at a distance  $L$  from the pivot point. If contestant 1 pushes the lever at distance  $a_{12}$  from the pivot, the force sent to the body will be  $X_1(a_{12}/L)$ . That force will be balanced by the frictional force  $(\mu X_2)$  generated by contestant 2, where  $\mu$  is the (unadjusted) coefficient of friction. The resulting force-balance equations for contestant 1 as attacker and for contestant 2 as attacker are

$$X_1(a_{12}/L) = \mu X_2 \quad \text{and} \quad X_2(a_{21}/L) = \mu X_1 .$$

The coefficients of friction are unknown and assumed to be equal. But we can rearrange the equations to form adjusted coefficients of friction in the following way:

$$X_1 = (\mu L/a_{12})X_2 \quad \text{and} \quad X_2 = (\mu L/a_{21})X_1$$

This is progress, but we still have a pair of equations which cannot be solved. This is where the reference force of magnitude  $F$  comes into play. We substitute it for the defending force and find a new point for the attacker to push against his lever in order to keep the body stationary. This point will be  $b_1$  for contestant 1 and  $b_2$  for contestant 2. New equations, similar to the last pair but with  $F$  as the defense

force, are

$$X_1 = (\mu L/b_1)F \text{ and } X_2 = (\mu L/b_2)F$$

Multiplying the two equations for  $X_1$  together leads to

$$(X_1)^2 = [(\mu L)^2 F / (a_{12} b_1)] X_2 = (A_{12}) X_2$$

where the entire expression inside the square brackets is taken as a single constant, given in the problem as an input scoreboard entry ( $A_{12}$ ). This is now the equation for the contest algorithm, with contestant 1 as attacker, in a two-person duel. Reversing the indices gives the equation for contestant 2 as attacker. Rewriting the equation slightly by multiplying top and bottom on the right side by  $X_2$  gives an equation for the squares of two forces in frictional balance. Now we can write the expression for the force balance when contestant 1 is attacking:

$$X_1 = \mu L [F / (X_2 a_{12} b_1)]^{1/2} X_2 = (A_{12} / X_2)^{1/2} X_2 = \mu_{12} X_2$$

The adjusted coefficient of friction for contestant 1 attacking contestant 2 is  $\mu_{12}$ , computed as shown. Notice that the defender's strength is an explicit part of the friction adjustment while the attacker's strength comes in through the leveraging process. It is plain to see now how a weak defender gets the large coefficient of friction he needs in his brake, since his strength is in the denominator of the adjustment factor.

The quantity  $\mu$  can be taken as 1. The symbol was kept in the equations until now to show how it is adjusted to give a new friction constant for each brake. Reversing the indices gives the other equation for the interacting pair and the pair can be easily solved for  $X_1$  and  $X_2$ . The adjusted coefficient of friction ( $\mu_{ij}$ ) has no further use and need not be retained.

A feature of this model is the possibility for self adjustment or “automatic tuning”. By connecting a cable via pulleys from the movable body to the (frictionless) sliding point on the lever where the attacking force is applied, the motion of the body can be used to slide the point closer to the pivot of the lever arm, reducing the driving force on the body. When the driving force is weak enough, the brake will stop the motion at the force-balance point..

### Beyond the Duel

To go from the simple duel to a contest involving more than two participants, I will first give the reasoning which led to the discovery (by me) of a form of the contest algorithm which has been used by the military for many years. In the analysis of military combat, the results of engagements involving many types of weapons are put into tables of fractional losses of weapon types produced by opposing weapon types--loss-by-cause tables or scoreboards. (*Fractional* losses are used so that all entries have a common dimensionality.) It seemed reasonable to expect that the scoreboard data could be processed in some way to produce numbers showing how the weapon systems in an engagement compare with each other. A system that kills a lot of important opposing systems should be important itself and should give a boost of importance to an opposing system that can kill it. Any process for comparing weapon systems must satisfy the requirement for *groupability*: the idea that splitting or combining groups of identical weapons (making more or fewer groups out of them) must make no difference in the importance that is computed for an individual weapon in any of the groups.

The most obvious approach to finding weapon-comparison numbers was to use a system of linear equations. I began trying this around 1968 and thought about it occasionally in connection with work I was doing, but I could not make it work. Others independently thought of the method and had the same lack of success. (It had to do with unreasonableness of solutions in response to smooth changes in inputs.) Then in the mid-80s I got into a problem area (combat modeling) that really needed weapon comparisons and I tried to find a system of *nonlinear* equations that satisfied the groupability criterion. To my surprise, I found by trial and error that a particular family of nonlinear equations that did this, and I accepted the member of the family that agreed with the linear method (described in the next paragraph) for the case of two participants.

The linear method for two participants has  $X_1 = kA_{12} X_2$  and  $X_2 = kA_{21} X_1$ , where  $k$  is a constant of proportionality which must have a certain value to permit a solution. (In this case, the constant needs to be  $(A_{12}A_{21})^{-1/2}$ .) Only relative  $X$  values are obtainable, and their ratio is  $(A_{12}/A_{21})^{1/2}$ . The square of this ratio gives the relative contributions of the two orthogonal  $X$  vectors to their vector sum or  $(X_1/X_2)^2 = (A_{12}/A_{21})$ . This agreement between the relative contributions and the ratio of the points in a scoreboard was deemed sufficient reason to say that the linear approach was correct for the duel, even if it could not be extended to more participants. (We are used to thinking of the comparison between players in terms of their scoring against each other.)

The family of nonlinear equations satisfying *groupability* is as follows:

$$(X_i)^n = \sum_j (A_{ij})^{n-1} f_{ij} X_j$$

where  $f_{ij} = A_{ij} / \sum_i A_{ij}$  and where  $n$  is an integer greater than 1. The subscript on the summation symbol shows which index is to be summed,  $i$  or  $j$ . (In words, the symbol  $f_{ij}$  stands for the fraction of all scores against defender  $j$  which are due to attacker  $i$ .)

When the  $n = 3$  member of the family is chosen the quantity  $(X_1/X_2)^2$  is again equal to  $(A_{12}/A_{21})$ , as with the linear method. That agreement seemed to dictate the use of the  $n = 3$  member in the equations of the contest algorithm. It was always necessary to assume the presence of some form of force adjustment with gearing or hydraulics in a mechanical-analog model of a contest, but no simple adjustment mechanism presented itself. This led me to reconsider the original choice of  $n = 3$  for the family member and the supposed need for agreement with the linear equations for the duel. The way a simple lever can be used for adjustment, as described above, leads me now to conclude that the  $n = 2$  member of the nonlinear family is the correct one.

So, where does that leave us with respect to the ratio of relative contributions to the vector-sum magnitude in a duel? That ratio is now  $(X_1/X_2)^2 = (A_{12}/A_{21})^{2/3}$  and it might be of interest to ponder why it is less extreme than the pure ratio of scores. The answer is that attack alone does not determine how the contestants compare. Bringing in opposing defense capabilities works against attack capabilities. Putting in some numbers might be a help here. Suppose the ratio of scores were 8:1. This makes the ratio of contributions 4:1, which might make a better comparison, since the weaker side deserves some credit for its defense against a strong opponent. At any rate, there is no *a priori* reason to expect the ratio of squared strengths to be equal to the ratio of scores. In fact, having now found  $(X_1/X_2)^2$  to be equal to  $(A_{12} X_2/A_{21} X_1)$ , we see that the ratio of *weighted* scores is more logical than the ratio of pure scores, and the defense capability is neatly brought in with its weight factor.

Having settled upon the correct form of the equations for determining strengths of contest participants, let us derive that set for contests larger than the duel. The general scheme is that of a set of force

vectors working in a vector space of n dimensions, where n is the number of participants in the contest. Every participant has two vectors, one for attack and the other for defense, both with the same magnitude but generally different in direction. Attackers and defenders use components of their strength vectors in each of their interactions (which are duels). Friction coefficients in individual duels are dynamic parts of the overall contest, increasing or decreasing in such a way that the contestants achieve force balance in each duel.

When contestant i is attacking contestant j in direction j, contestant j is using brake force perpendicular to direction j, or in direction i. Thus the two engaged contestants point strength-vector components at each other, whether attacking or defending. This whole n-dimensional picture is purely a mathematical construct; a machine of this sort is imaginary, but its parts can be oriented in any way in real space. All that is needed to extend the two-person contest equation to the general case (any number of contestants) is to insert a factor for the direction of the defender's vector. The fractions  $f_{ij}$  serve that purpose, since they sum (over all attackers) to unity and therefore act as squares of direction cosines. (Direction cosine squares sum to unity for vector spaces of any number of dimensions.) One square root of the fraction is the direction cosine for the defense vector and a second square root is for the reference-force vector (whose magnitude is contained within the scoreboard entry but whose direction is parallel to the defense vector). When the fraction is inserted, the equation for the general case becomes a sum over individual duels, as follows:

$$(X_i)^2 = \sum_j (A_{ij}) f_{ij} X_j \quad \text{where} \quad f_{ij} = A_{ij} / \sum_i A_{ij}$$

This is the  $n = 2$  member of the family that was found by trial and error to satisfy groupability, and the derivation is complete. These equations are easy to solve by iteration, even for very large problems.

The basic contest equation gives the square of the magnitude of the strength vector of a contestant, and each term in the sum is the square of an attack-vector component in one of the directions of the vector space. Therefore we can immediately write the vector form for the attacker as

$$\underline{X}_i = \sum_j [(A_{ij}) f_{ij} X_j]^{1/2} \cdot \underline{u}_j$$

where the underline designates a vector quantity and  $\underline{u}_j$  is a unit vector in the j direction. Both i and j range from 1 to the number of contestants in the problem. Solving the set of coupled nonlinear equations which make up the algorithm tells us everything there is to know about the set of vectors--magnitudes, attack directions from the vector form just shown, and defense directions from the square roots of the fractions  $f_{ij}$ . Every i,j pair represents an individual duel with attack and defense vectors that have been determined from the scoreboard alone.

Most applications make use of the *contest vector*, defined as the vector sum of all attack vectors in the contest. With a little manipulation of the equations, it can be easily shown that the square of the magnitude of the contest vector is  $C^2 = \sum_{ij} A_{ij} X_j$ . The summation is over both i and j. Dividing through by  $C^2$  gives the fractional contribution of each attacker/defender interaction (duel) to the overall intensity of the contest. The summation is over all elements of a new matrix formed by attaching weights of defenders to the original contest scores. This new matrix is scalar, but we had to solve nonlinear equations in a vector space to find the participants' strengths to be used as weights. Since index i refers to the attacker, we can use the partial sum of  $C^2$  (over j alone) for any attacker i, to obtain the *applied strength* of participant i. (The applied strength is a scalar: the contribution of the participant's strength vector to the contest vector.) To compare athletic entities (individuals or teams) on

the basis of wins and losses, the applied strengths should be used instead of the raw scoreboard entries. The elegant simplicity of the  $C^2$  result is another reason to conclude that we now have the correct form of the equations for contest analysis.

We can find the angles which the attack vectors make with each other, and this leads into the subject of *enmity* and *amity* in the contest (amity being mutual support). In a duel, it is obvious that enmity must be 100% and amity zero, and it would follow that the right angle between attack vectors in the duel corresponds to full enmity. Full amity could then be nothing other than a zero angle between attack vectors. The cosine of the angle between attack vectors is a logical designator for degree of amity. Only the attack vectors figure in the specification of amity between participants--or in determinations of relative contributions to the contest of any sort--because defense is passive. Defense is not needed by a contestant that is not scored against (attacked), and the direction of the defense vector is indeterminate for that case.

Two contestants which are identical in their behavior with respect to every other contestant must have vectors that are equal in magnitude and parallel for attack and again for defense. Contestants having proportional scores against each of their opponents (*rows* of the data array *proportional* to each other) will have parallel attack vectors. Identical behavior of two contestants in the defense mode means that their scoreboard entries are identical for each of their attackers, causing  $f_{ij}$  to be identical for each  $i$  (i.e., *columns* of the data array *equal* to each other--not proportional). When attack vectors of two contestants are parallel, and defense vectors also, the two contestants are *groupable* and can be replaced with one of combined strength. (Groupability was the reason for the selection of these contest equations for combat analysis in the beginning.) Contestants with parallel vectors in only one of the two modes will not be groupable.

The last paragraph explains why scores in contests must be in the form of *score per defender* whenever grouping is a consideration. Under grouping, the scoreboard entries for attack (rows) are added together, and one of the (equal) columns for defense is deleted each time a group is formed from two subgroups. This arrangement was automatically satisfied in the original application of this method (combat modeling), because the scores were fractional losses of combatants--and a fractional loss is the same, no matter how many are in the group. It now is clear that all contest situations require the scores to be on a per-defender basis in order that the columns of the score matrix (the scores indexed to different attackers for particular defenders) do not change when defenders are grouped.

At this point, we have a complete imaginary model constructed solely from physical concepts, with strengths of participants and all angular quantities calculable from scoreboard entries alone. Variables describing the operation are lumped together into the scoreboard entries. No theoretical difficulty arises by resorting to an imaginary machine; indeed, the progress of science is studded with "thought experiments" used to illustrate complex points of theory. It should be pointed out that the imaginary nature of this machine has to do with practical considerations alone and not with operation in  $n$  dimensions. The directions of potential motion in the machine are given no particular spatial orientation in the real world.

The proof that the system of equations of this paper constitutes a valid description for a contest is in the physical and mathematical description of the machine itself. The only assumption of the method is that contest scores are related to friction constants. This assumption is quite reasonable, from a qualitative standpoint, since a large contest score identifies a weak defender who cannot put much force upon his brake and has to rely upon more effective friction material to stop the attacker.

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